Conclusions

An approximate method for computing near-optimal, minimum-fuel, planar lunar trajectories for low-thrust spacecraft has been developed. Our method approximates the long-duration powered Earth-escape and moon-capture spirals with curve fits from universal low-thrust trajectory solutions and numerically computes the translunar coasting trajectory between the curve-fit boundaries using the restricted three-body problem dynamics. The approximate method requires only four optimization variables, and solutions are readily obtained by using sequential quadratic programming. Near-optimal solutions for a wide range of initial thrust-to-weight ratios and initial and terminal circular orbit altitudes are obtained both quickly and with little computational load. The performance and orbit transfer characteristics of the near-optimal solutions exhibit a very close match with published optimal lunar trajectory solutions. This method would be a useful preliminary design tool for spacecraft and mission designers.

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Analytical Missile Guidance Laws with a Time-Varying Transformation

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I. Introduction

THE most popular homing missile guidance is based on a control law called proportional navigation.[1] The basic notion is that, if the line-of-sight rate is annulled, then (for a nonmaneuvering, constant-velocity target) the missile is on a collision course. If the target is considered smart or maneuvering, variations in the proportional navigation have been shown to result in better miss distances. These variations have been given optimal control foundations through linear quadratic Gaussian formulations.[2–5]

There are, however, a few problems with the use of such guidance laws. First, the measurements in an end game are nonlinear (bearing angle, range, and range rate) in Cartesian coordinates. As a consequence, there is linearization in the filtering update process. The measurements are linear in a polar coordinate–based state space. However, the propagation between the measurement updates in this case leads to nonlinear equations. Therefore, the states used in the guidance law are suboptimal. The second problem lies with the guidance law, which is formulated assuming separability of the guidance (control) law, and the estimators, which do not hold. It is usually formulated in Cartesian coordinates for linearity.[2,4] As a result, there is considerable scope for research in improving the missile performance in terms of estimator, guidance, and autopilot in an intercept scenario.[5]

This study is focused on obtaining improvements with a properly posed controller for guidance. Such a view will enable us to integrate the estimator in the loop in an optimal way. The central idea here is that the polar coordinates present a natural coordinate system for a missile engagement. A few papers in the literature[6,7] deal with guidance laws in polar coordinates, but their formulations do not derive an optimal guidance law as they do in this study. Lin and Tsai[8] and Rao,[9] in an extension of Ref. 8, consider an optimal control formulation seeking to maximize the final missile speed. In this study, we seek to minimize the control energy, and we consider a time-varying weight on the control to shape the relative kinematics. To obtain a closed-form solution for the commanded accelerations, the radial and transverse coordinates are decoupled. The decoupling of the coordinates leads to a two-point boundary-value problem with linear, time-varying coefficients. However, with a time-varying transformation, a class of closed-form solutions are obtained that yield several proportional guidance laws.

The rest of this Note is organized as follows: The optimal guidance problem is developed in polar coordinates in Sec. II. This problem is further shown to decompose into two decoupled optimal control problems, where a closed-form control solution is available in the radial direction and a time-varying linear dynamic system has to be solved for control in the transverse direction. A commonly used approximation for time-to-go and a transformation are shown to lead to a class of proportional navigation–type feedback guidance laws in Sec. III. The conclusions are summarized in Sec. IV.

II. Optimal Guidance in Decoupled Polar Coordinates

The dynamics of a target-intercept geometry are expressed by a set of coupled nonlinear differential equations in an inertial polar coordinate system as (Fig. 1)

$$\ddot{r} - r\dot{\theta}^2 = a_{T_r} - a_{M_r} \tag{1}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_{T_\theta} - a_{M_\theta} \tag{2}$$

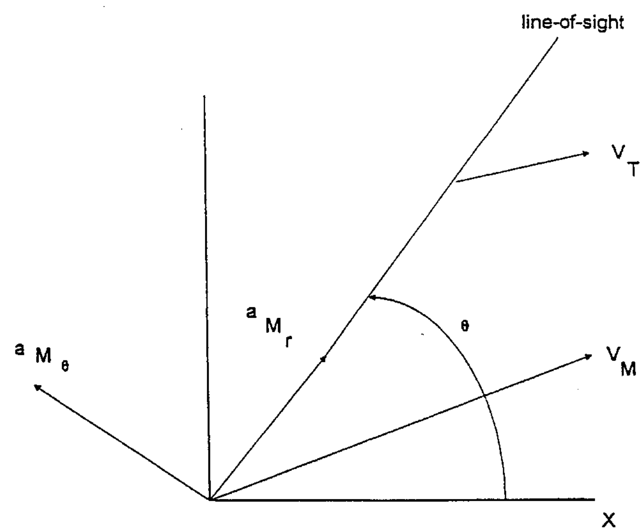

Fig. 1 Engagement geometry.

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In Eqs. (1) and (2), r is the relative range between the target and the missile, θ is the bearing angle, and a_{T_r} and a_{M_r} are, respectively, the target and missile accelerations in the line-of-sight (radial) direction. Similarly, a_{T_θ} and a_{M_θ} represent the target and missile accelerations, respectively, in the transverse directions. Dots denote differentiations with respect to time. Note that, if the analysis is carried out in three dimensions, there will be another equation involving elevation angle.

Line-of-Sight (Radial) Commanded Acceleration

It can be easily observed that Eqs. (1) and (2) are coupled. To decouple the dynamics, a pseudocontrol in the radial direction, $a_{M_{r1}}$, is defined as

$$a_{M_{r1}} = a_{M_r} - r\dot{\theta}^2 \tag{3}$$

This definition decouples the radial coordinate from the transverse coordinate. It facilitates a state space y in the line-of-sight direction as $y = [r, \dot{r}, a_{T_r}]^T$ and describes their dynamics as

$$\dot{y}_1 = y_2 \tag{4}$$

$$\dot{y}_2 = y_3 - a_{M_{r1}} \tag{5}$$

$$\dot{y}_3 = -\lambda_r y_3 \tag{6}$$

where λ_r is the inverse time constant associated with target acceleration.

The optimal guidance law in the radial direction is obtained as a solution to minimizing the performance index J_r, where

$$J_r = \frac{1}{2} S_{f_r} y_{1_f}^2 + \frac{1}{2} \int_0^{t_f} \gamma a_{M_{r1}}^2 \, \mathrm{d}t \tag{7}$$

In Eq. (7), y_{1_f} is the value of the relative range (miss distance) at the final time t_f, S_{R_f} is the weight on the miss distance, and γ is the weight on the pseudocontrol effort. The final time t_f, which is the time-to-go, is approximated as $r/|\dot{r}|$. The minimizing control is

$$a_{M_{r1}}(t) = (t_f/\gamma)\lambda_1(t) \tag{8}$$

and

$$\lambda_1(t) = S_{f_r}\big(y_1(t) + t_f y_2(t) + \big(1/\lambda_r^2\big)a_{T_r} \times [\exp(-\lambda_r t_f) + \lambda_r t_f - 1]\big)\big/\big(1 + t_f^3 S_f/3\gamma\big) \tag{9}$$

In Eqs. (8) and (9), λ_1 is a Lagrangian multiplier that adjoins the state in Eq. (5) to the performance index in Eq. (7). The actual missile acceleration can be obtained from Eq. (3) as

$$a_{M_r}(t) = a_{M_{r1}}(t) + r(t)\dot{\theta}^2(t)$$

The instantaneous values of the relative range $r(t)$ and relative range rate $\dot{r}(t)$ can be solved for by integrating Eqs. (4–6).

Transverse Acceleration

The equation of motion in the transverse direction in Eq. (2) can be rewritten as

$$\ddot{\theta} = (2\dot{r}\dot{\theta}/r) + (1/r)a_{T_\theta} - (1/r)a_{M_\theta} \tag{10}$$

Note that, since r and $\dot{r}$ are known from Eqs. (4–6), they can be treated as functions of time. Consequently, Eq. (10) can be expressed as a time-varying linear differential equation as

$$\ddot{\theta} = f(t)\dot{\theta} + g(t)a_{T_\theta} - g(t)a_{M_\theta} \tag{11}$$

where $f(t) = -(2\dot{r}/r)$ and $g(t) = -(1/r)$.

With first-order dynamics for target acceleration, Eq. (11) can be expressed in a state space $z = [\theta, \dot{\theta}, a_{T_\theta}]^T$ as

$$\dot{z}_1 = z_2 \tag{12}$$

$$\dot{z}_2 = f(t)z_2 + g(t)z_3 - g(t)a_{M_\theta} \tag{13}$$

and

$$\dot{z}_3 = -\lambda_\theta z_3 \tag{14}$$

where λ_θ is the inverse time constant associated with the transverse target acceleration.

A performance index J_θ, similar to Eq. (7) for the transverse direction, is

$$J_\theta = \frac{1}{2} S_{f\theta} z_{2_f}^2 + \frac{1}{2} \int_0^{t_f} \big(\gamma_1 z_2^2 + \gamma_2 a_{M_\theta}^2\big) \, \mathrm{d}t \tag{15}$$

where $S_{f\theta}$, γ_1, and γ_2 are the weights.

The optimization process to yield the controller minimizing Eq. (15) leads to a two-point boundary-value problem:

$$\begin{bmatrix} \dot{z}_2 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} f(t) & -g^2(t) \\ -\gamma_1 & -f(t) \end{bmatrix} \begin{bmatrix} z_2 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} g(t) z_{30} \exp(-\lambda_\theta t) \\ 0 \end{bmatrix} \tag{16}$$

where z_{30} is assumed known and $\lambda_{2_f} = S_{f\theta} z_{2_f}$. In Eq. (16), λ_2 represents the Lagrangian multiplier that adjoins Eq. (13) to the performance index J_θ. This system can be solved either numerically by techniques such as the shooting method or analytically if functional forms of $f(t)$ and $g(t)$ are known. The minimizing control in the transverse direction is given by

$$a_{M_\theta}(t) = \lambda_2(t) g(t)/\gamma_2 \tag{17}$$

III. Class of Proportional Navigation Guidance Laws Through Transformations

Although Eqs. (8) and (17) represent a general decoupled solution, interesting analytical solutions for the terminal guidance problem and a feedback guidance law can be obtained through a time-varying transformation of coordinates like the one used in Ref. 10. For comparison with existing results, the target acceleration is assumed zero.

Note that the final time (time-to-go) calculation involves an assumption that the closing velocity (relative range rate) is constant. This assumption can be translated to

$$r(t) = \dot{r}(t_f - t) \tag{18}$$

where t is the current time. In a feedback rule, this assumption is not very restrictive because $\dot{r}$ is updated at each instant. By using Eq. (18) in Eq. (10), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta} = \frac{2\dot{\theta}}{t_f - t} - \frac{a_{M_\theta} t_f}{r_0(t_f - t)} \tag{19}$$

This equation is difficult to integrate numerically because $(t_f - t)$ appears in the denominator. Hence, define a variable u as

$$u = (t_f - t)^2 \dot{\theta} \tag{20}$$

The differential equation for u (after some algebra) is

$$\dot{u} = -a_{M_\theta}(t_f/r_0)(t_f - t) \tag{21}$$

Note that Eq. (21) is devoid of any expression in u on the right-hand side and $(t_f - t)$ in the denominator.

The optimal control problem is now solved through the use of the new variable u. Consider a performance index $J_{\theta 1}$, given by

$$J_{\theta 1} = \int_0^{t_f} \frac{1}{2} \gamma_2(t) a_{M_\theta}^2 \, \mathrm{d}t \tag{22}$$

Although this performance index seems simpler than J_θ in Eq. (15), it can be shown that Eq. (22) can accommodate a variety of designs by assuming different functional representations for the weight $\gamma_2(t)$.

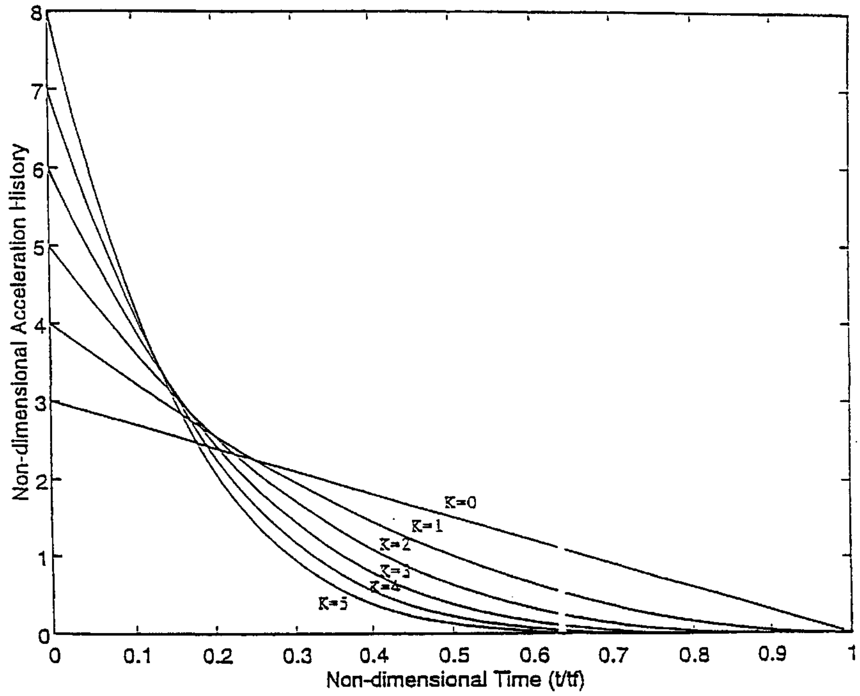

Fig. 2 Nondimensional acceleration history.

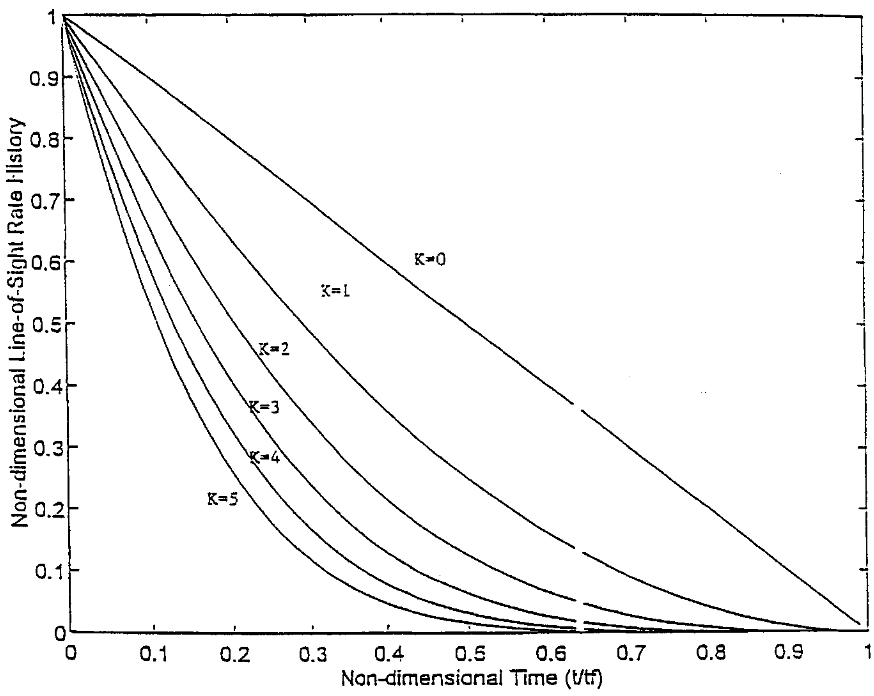

Fig. 3 Nondimensional line-of-sight rate history.

The Hamiltonian H of this system is given by

$$H = \frac{1}{2}\gamma_2(t)a_{M_\theta}^2 - \lambda a_{M_\theta}(t_f/r_0)(t_f - t) \tag{23}$$

The propagation of the Lagrangian multiplier λ is governed by

$$\dot{\lambda} = 0 \tag{24}$$

Hence, λ is a constant. The optimality condition leads to

$$a_{M_\theta} = \frac{t_f(t_f - t)}{r_0\gamma_2(t)}\lambda \tag{25}$$

By using Eq. (25) in the propagation equation for u in Eq. (21), we get

$$\dot{u} = \frac{-t_f^2(t_f - t)^2}{r_0^2}\frac{\lambda}{\gamma_2(t)} \tag{26}$$

We will now derive a family of proportional navigation laws. Let

$$\gamma_2(t) = (t_f - t)^{-K} \tag{27}$$

where K is a positive integer. The implication of this time-varying weight is that the control effort should achieve most of the trajectory shaping before the time-to-go reaches the last second.

With this expression for $\gamma_2(t)$, Eq. (26) can be integrated. The Lagrangian multiplier consequently can be solved for as

$$\lambda = \frac{r_0^2 \dot{\theta}_0 (K+3)}{t_f^{K+3}} \tag{28}$$

With Eq. (28), we can solve for $u(t)$ from Eq. (26) as

$$u(t) = u(0) + \frac{\dot{\theta}_0}{t_f^{K+1}} \left[(t_f - t)^{K+3} - t_f^{K+3} \right] \tag{29}$$

The control acceleration $a_{M_\theta}(t)$ and the line-of-sight rate $\dot{\theta}(t)$ can be obtained as explicit functions of time as

$$a_{M_\theta}(t) = (K+3)\dot{r}_0\dot{\theta}_0[1 - (t/t_f)]^{K+1}, \qquad K \neq -3 \tag{30}$$

and

$$\dot{\theta}(t) = \dot{\theta}_0[1 - (t/t_f)]^{K+1} \tag{31}$$

By varying K ($\neq -3$), we can obtain a family of proportional navigation guidance laws. In particular, let $K = 0$ in Eqs. (30) and (31). We get

$$a_{M_\theta}(t) = 3\dot{r}_0\dot{\theta}_0[1 - (t/t_f)] \tag{32}$$

and

$$\dot{\theta}(t) = \dot{\theta}_0[1 - (t/t_f)] \tag{33}$$

The plots of nondimensionalized acceleration variations and nondimensionalized line-of-sight rate variations with nondimensionalized time for different K are presented in Figs. 2 and 3, respectively. It can be seen that, as K increases, the initial line-of-sight rates are brought down quickly with higher levels of acceleration. In contrast, intermediate values of acceleration allow initial lines-of-sight to remain fairly high (good for the estimator) and then drive the line-of-rate to zero to effect the intercept. If K is low ($n = 0$), higher levels of accelerations are required, even near the end.

IV. Conclusions

A class of proportional navigation guidance laws has been derived through an approximation of time-to-go and a transformation of state variables.

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Linear Quadratic Pursuit–Evasion Games with Terminal Velocity Constraints

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I. Introduction

MODERN guidance techniques are based mainly on optimal control theory and on differential games. Reference 1 compares these methods and advocates the use of differential game guidance methods, particularly against maneuvering targets. The widely used proportional navigation is in itself an optimal strategy in which the cost is the missile's control effort and a zero miss (against a nonmaneuvering target) is imposed by the boundary conditions. In optimal rendezvous[2] the same cost is employed with the additional constraint of zero terminal lateral velocity (with respect to a nominal line-of-sight). For some cases, this additional requirement may be of importance from an operational or technical point of view. In a well-known differential game (originally solved in Ref. 3) the target is maneuvering to maximize a weighted sum of miss distance, the missile's control effort, and its own control effort (the latter with negative weight). The pursuer is trying to minimize the same cost function (a zero-sum game). The strategy obtained by this classical work will be referred to as the game-theoretic optimal intercept.

The present work is a natural extension of the last two cases. The differential-game approach is used where the terminal lateral velocity is introduced into the cost in addition to the adversaries control efforts and the miss distance. The case of zero terminal conditions will naturally be called the game-theoretic optimal rendezvous

This Note is organized as follows. After introducing the governing equations, the general problem will be formulated. The problem will be solved in closed form and the above-mentioned cases of one-sided optimal rendezvous and game-theoretic optimal intercept will be shown to be special cases of this problem. We will then concentrate on perfect (zero-miss) intercepts with terminal velocity constraints and, in particular, on the game-theoretic optimal rendezvous for which optimal strategies as well as the resulting optimal trajectories will be given in closed form.

II. Problem Formulation

We make the following assumptions:

1) The pursuit–evasion conflict is two dimensional, in the horizontal plane.

2) The speeds of the pursuer P and the evader E are constant.

3) The trajectories of P and E can be linearized around their collision course.

4) Both opponents can directly control their lateral accelerations.

5) The pursuer is more maneuverable than the evader.

Under these assumptions the problem has the following state-space representation[2]:

$$\dot{X} = AX + Bu + Dw$$
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1}$$

where x_1 and x_2, the components of X, are the relative displacement and velocity and u and w are the normal accelerations of the pursuer and the evader, respectively.

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